

Problem for the whole class:

Consider the following bead moves:

- a) $\bullet \circ \rightarrow \circ \bullet$ (swap two beads if they are $\bullet \circ$)
- b) $\bullet \bullet \rightarrow$ (nothing) (remove two $\bullet$ beads)
- c) $\circ \circ \circ \rightarrow$ (nothing) (remove three $\circ$ beads)

What is the smallest bead sequence you can make using moves a), b), c) starting from:

- 1. $\circ \circ \bullet \circ \bullet \circ \bullet \circ \bullet \circ$ can remove all
- 2. $\circ \circ \bullet \circ \bullet \circ \bullet \circ \bullet \circ \bullet$ $\bullet$ remains
- 3. $\circ \circ \bullet \circ \bullet \circ \bullet \circ \bullet \circ \circ$ $\circ$ remains

We can use $\bullet \circ \rightarrow \circ \bullet$ to move all the black beads to the right, then remove a multiple of 2 black beads using $\bullet \bullet \rightarrow$ and a multiple of 3 white beads with $\circ \circ \circ \rightarrow$.

Note all operations preserve mod 2 of black and mod 3 of white, so the $\bullet$ and $\circ$ can't be removed in 2/3. Since we start with 1 mod 2/3.

What if we add $\rightarrow \circ \circ \circ \circ$?

We can use this to fix mod 3 of white:

$$\underset{\text{new}}{\circ} \xrightarrow{\quad} \circ \circ \circ \circ \xrightarrow{c} \underset{\text{new}}{\circ \circ} \xrightarrow{\quad} \circ \circ \circ \circ \circ \circ \xrightarrow{c} \circ \circ \circ \xrightarrow{c}$$

Maybe surprising: adding beads can help remove!

What if we add $\rightarrow \bullet \bullet \bullet \bullet$?

Mod 2 of black is still preserved, so this one does not help! even with this move $\bullet$ remains in 2.